$\text{boy } V^* = \text{boy } V$

$V^* = V$ üzerinde lineer formlar.

Teorem 5: V bir vektör uzayı ve $S = \{u_1, \ldots, u_n\}$ V nin bir sıralı bazı olsun. Aşağıda tanımı verilen $\{f_1, f_2, \ldots, f_n\} \subseteq V^*$ kümesi V^* için bir bazdır.

$f_1(u_1) = 1, \quad f_1(u_j) = 0 \quad (j \neq 1 \text{ için})$

$f_2(u_2) = 1, \quad f_2(u_j) = 0 \quad (j \neq 2 \text{ için})$

$\vdots$

$f_n(u_n) = 1, \quad f_n(u_j) = 0 \quad (j \neq n \text{ için})$

Özetle: $f_i(u_j) = \delta_{ij}$

İspat: 1) $\{f_1, \dots, f_n\}$ lineer bağımsızdır:

$$a_1 . f_1 + a_2 f_2 + \dots + a_n f_n \equiv 0$$

Her bir j için

$$(a_1 f_1 + \dots + a_n f_n)(u_j) = 0$$

$$\underbrace{a_1 . f_1(u_j) + \dots +}_{0} a_j . f_j(u_j) + \underbrace{\dots + a_n f_n(u_j)}_{0} = 0$$

$$a_j . \underbrace{f_j(u_j)}_{= 1} = 0 \Rightarrow a_j = 0$$

Bu argüman her $1 \leq j \leq n$ için tekrarlanabilir;

$a_1 = a_2 = \dots = a_n = 0 \Rightarrow$ Lineer Bağımsızlık ✓

boy $V = n$ $(S=\{u_1, \dots, u_n\}$ bazı$)$

boy $V^* =$ boy $V = n$

$\{f_1, \dots, f_n\}$ n elemanlı lineer bğ.sız bir küme ; n boyutlu uzayın (V^*) baz olur.

Tanım: V bir vektör uzayı, $S=\{u_1, \dots, u_n\}$ bir sıralı bazı olsun. V^* uzayının yukarıda tanımlanan $S^* = \{f_1, f_2, \dots, f_n\}$ bazına S'in dual bazı denir

$(f_i(u_j) = \delta_{ij})$

Teorem 6: V sonlu boyutlu bir vektör uzayı, $S=\{u_1, \ldots, u_n\}$ bir sıralı bazı ve $S^*=\{f_1, \ldots, f_n\}$ V^* ın S e dual olan bazı olsun. V üzerindeki her f lineer formu için

$$f = \sum_{i=1}^{n} f(u_i) \cdot f_i \qquad \text{dir.}$$

V daki her v vektörü için de

$$v = \sum_{i=1}^{n} f_i(v) u_i$$

geçerlidir.

İspat: $f \in V^*$ ve

$S^* = \{f_1, \dots, f_n\}$ için

$f = c_1 \cdot f_1 + \dots + c_n \cdot f_n$

$1 \leq j \leq n$ için $f(u_j) = c_1 \cdot f_1(u_j) + \dots + c_j \cdot f_j(u_j) + \dots$
$+ c_n f_n(u_j)$

$f(u_j) = c_j \cdot \underbrace{f_j(u_j)}_{1}$ $\left(f_i(u_j) = \delta_{ij} \right)$

$c_j = f(u_j) \Rightarrow f = \sum_{j=1}^{n} f(u_j) \cdot f_j$ ✓

$v \in V$ alalım $S = \{v_1, \dots, v_n\}$ olsun.

$$v = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$$

$1 \le j \le n$, $$f_j(v) = a_1 . f_j(v_1) + \dots + \underbrace{a_j . f_j(v_j)} + \dots + a_n . f_j(v_n)$$

$$f_j(v) = a_j . \underbrace{f_j(v_j)}_{1}$$

$$a_j = f_j(v)$$

$$\Rightarrow \quad v = \underbrace{f_1(v)} . v_1 + \underbrace{f_2(v)} . v_2 + \dots + \underbrace{f_n(v)} . v_n$$

[$\{f_1, \dots, f_n\}$ bazına V nin koordinat formları da denir. (S bazına göre)]

Hatırlatma: $S=\{u_1,u_2,...,u_n\}$ V nin bir sıralı bazı, $T=\{f_1,f_2,...,f_n\}$ V^* ın S e dual olan bazı olsun.

$f\in V^*$ aldığımızda

f in V deki S bazına ve $\mathbb{R}$ nin $\{1\}=T$ bazına göre temsilcisi:

$c=c\cdot 1$

$$A=\left[\,[f(u_1)]_T \;\vdots\; [f(u_2)]_T \;\vdots\; \cdots \;\vdots\; [f(u_n)]_T\,\right]$$

$$=\left[\, f(u_1) \;\vdots\; f(u_2) \;\vdots\; \cdots \;\vdots\; f(u_n)\,\right]$$

$f(u_i)=b_i$

$$A=[\,b_1\;\; b_2\;\; \cdots\;\; b_n\,]$$

Her $v \in V$ için $v = x_1 . u_1 + x_2 . u_2 + \dots + x_n . u_n$

yazılışı var;

$$f(\underline{\underline{v}}) = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}}_{[v]_S} = \underline{b_1 x_1 + b_2 x_2 + \dots + b_n x_n}$$

Örnek: $\mathbb{R}^2(V)$ nin $S=\{\underbrace{(1,2)}_{u_1},\underbrace{(3,1)}_{u_2}\}$ bazının duali S^* hangi kümedir?

$$V^*=\{\underline{ax+by} \mid a,b\in\mathbb{R}\}$$

$$S^*=\{f_1,f_2\} \quad \left(\text{boy}(\mathbb{R}^2)^*=\text{boy}(\mathbb{R}^2)\right)$$

$f_1(x,y)=a_1x+b_1y$ (Hatırlatmada verilen form)

$f_2=a_2x+b_2y$

$f_1(u_1)=\underline{1}$, $f_1(u_2)=0$ ve $\left(f_i(u_j)=\delta_{ij}\right)$

$f_2(u_1)=0$, $f_2(u_2)=\underline{1}$

$$f_1(1,2) = a_1 \cdot 1 + b_1 \cdot 2 = 1$$
$$f_1(\underbrace{3,1}_{v_2}) = a_1 \cdot 3 + b_1 \cdot 1 = 0$$
$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 1 & 0 \end{array}\right]$$

$$f_2(\overbrace{1,2}^{u_1}) = a_2 \cdot 1 + b_2 \cdot 2 = 0$$
$$f_2(\underbrace{3,1}_{u_2}) = a_2 \cdot 3 + b_2 \cdot 1 = 1$$
$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 1 & 1 \end{array}\right]$$

$$\begin{array}{cc} & \begin{array}{cccc} & & f_1 & f_2 \end{array} \\ & \left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right] \end{array} \overset{-3S_1+S_2}{\sim} \left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 0 & -5 & -3 & 1 \end{array}\right]$$

$$-\frac{1}{5} \cdot S_2 \sim \left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{5} & -\frac{1}{5} \end{array}\right] \overset{-2S_2+S_1}{\sim} \begin{array}{c} \left[\begin{array}{cc|c|c} 1 & 0 & -\frac{1}{5} & \frac{2}{5} \\ 0 & 1 & \frac{3}{5} & -\frac{1}{5} \end{array}\right] \\ \begin{array}{cccc} & & f_1 & f_2 \end{array} \end{array}$$

$$f_1(x,y) = -\frac{x}{5} + \frac{3y}{5}$$

$$f_2(x,y) = \frac{2x}{5} - \frac{y}{5}$$

Sağlama: $f_1(\underbrace{1,2}_{v_1}) = -\frac{1}{5} + \frac{3 \cdot 2}{5} = 1$

$$\left[\begin{array}{cc|c|c} 1 & 0 & -1/5 & 2/5 \\ 0 & 1 & 3/5 & -1/5 \end{array}\right]$$

$$\qquad\qquad\quad f_1 \qquad f_2$$

$$f_1(3,1) = -\frac{3}{5} + \frac{3}{5} = 0$$

$f_2(1,2) = \frac{2}{5} - \frac{2}{5} = 0$, $f_2(3,1) = \frac{6}{5} - \frac{1}{5} = 1$ ✓

Örnek: $\mathbb{R}^3$ ün $v_1=(1,-1,3)$, $v_2=(0,1,-1)$, $v_3=(0,3,-2)$ vekt. den oluşan bazının dualini bulunuz.

$(\mathbb{R}^3)^*$ için S e dual olan $S^*=\{f_1, f_2, f_3\}$

bazı:

$f_1(v_1)=1$, $f_1(v_2)=f_1(v_3)=0$

$f_2(v_2)=1$, $f_2(v_1)=f_2(v_3)=0$

$f_3(v_3)=1$, $f_3(v_1)=f_3(v_2)=0$

V^* ın her f vektörü $f(x,y,z)=a_1x+a_2y+a_3z$ şeklindedir.

$$f_1(x,y,z) = a_1 x + a_2 y + a_3 z$$
$$f_2(x,y,z) = b_1 x + b_2 y + b_3 \cdot z$$
$$f_3(x,y,z) = c_1 x + c_2 y + c_3 \cdot z \qquad \text{olsun:}$$

$$\left.\begin{array}{l} f_1(1,-1,3) = a_1 - a_2 + 3a_3 = 1 \\ f_1(0,1,-1) = a_2 - a_3 = 0 \\ f_1(0,3,-2) = 3a_2 - 2a_3 = 0 \end{array}\right\} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -2 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} f_2(1,-1,3) = b_1 - b_2 + 3b_3 = 0 \\ f_2(0,1,-1) = b_2 - b_3 = 1 \\ f_2(0,3,-2) = 3b_2 - 2b_3 = 0 \end{array}\right\} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -2 & 0 \end{array}\right]$$

$$\begin{array}{c}
\left[\begin{array}{ccc|c|c|c}
1 & -1 & 3 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & 0 \\
0 & 3 & -2 & 0 & 0 & 1
\end{array}\right] \\
\begin{array}{cccccc} & & & f_1 & f_2 & f_3 \end{array}
\end{array}
\underset{\sim}{-3S_2+S_3}
\left[\begin{array}{ccc|c|c|c}
1 & -1 & 3 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -3 & 1
\end{array}\right]$$

$$\left[\begin{array}{ccc|c}
1 & -1 & 3 & 0 \\
0 & 1 & -1 & 1 \\
0 & 3 & -2 & 0
\end{array}\right]$$

$$\begin{array}{c} -3S_3+S_1 \\ \sim \\ S_3+S_2 \end{array}
\left[\begin{array}{ccc|c|c|c}
1 & -1 & 0 & 1 & 9 & -3 \\
0 & 1 & 0 & 0 & -2 & 1 \\
0 & 0 & 1 & 0 & -3 & 1
\end{array}\right]
\underset{\sim}{S_2+S_1}
\left[\begin{array}{ccc|c|c|c}
1 & 0 & 0 & 1 & 7 & -2 \\
0 & 1 & 0 & 0 & -2 & 1 \\
0 & 0 & 1 & 0 & -3 & 1
\end{array}\right]$$

$f_1(x,y,z) = x$

$f_2(x,y,z) = 7x - 2y - 3z$

$f_3(x,y,z) = -2x + y + z$

$(1,-1,3),\ (0,1,-1),\ (0,3,-2)$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 7 & -2 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & -3 & 1 \end{array}\right]$$

$\downarrow$

f_1

$S^* = \{f_1, f_2, f_3\}$, S in dual bazıdır

Örnek: $V = P_1$ ve $\phi_1 : P_1 \to \mathbb{R}$

$$p(t) \mapsto \int_0^1 p(t)\,dt$$

$\phi_2 : P_1 \to \mathbb{R}$ olmak üzere;

$$p \mapsto \int_0^2 p(t)\,dt$$

$S^* = \{\phi_1, \phi_2\} \subseteq V^* = P_1^*$ kümesi (bazı) V nin hangi bazının dualidir?

$S = \{a_1 t + a_2,\ b_1 t + b_2\}$ V nin sözkonusu bazı olsun.

$$\phi_1(a_1t+a_2)=1 \qquad \text{ve} \qquad \phi_2(a_1t+a_2)=0$$

$$\phi_1(b_1t+b_2)=0 \qquad\qquad \phi_2(b_1t+b_2)=1$$

$$\phi_1(a_1t+a_2)=\int_0^1 (a_1t+a_2)\,dt=\frac{a_1}{2}t^2+a_2.t\Big]_0^1$$

$$\left.\begin{aligned}\phi_1(a_1t+a_2)&=\frac{a_1}{2}+a_2=1\\ \phi_1(b_1t+b_2)&=\frac{b_1}{2}+b_2=0\end{aligned}\right\}$$

$$\phi_2(a_1t+a_2)=\int_0^2 (a_1t+a_2)\,dt=\frac{a_1}{2}.t^2+a_2t\Big]_0^2=$$

$$=2a_1+2a_2=0$$

$$\phi_2(b_1t+b_2)=2b_1+2b_2=1$$

$$\left.\begin{aligned} \frac{a_1}{2} + a_2 &= 1 \\ 2a_1 + 2a_2 &= 0 \end{aligned}\right\} \qquad \left[\begin{array}{cc|c} \frac{1}{2} & 1 & 1 \\ 2 & 2 & 0 \end{array}\right]$$

$$\left.\begin{aligned} \frac{b_1}{2} + b_2 &= 0 \\ 2b_1 + 2b_2 &= 1 \end{aligned}\right\} \qquad \left[\begin{array}{cc|c} \frac{1}{2} & 1 & 0 \\ 2 & 2 & 1 \end{array}\right]$$

$$\begin{array}{c} \left[\begin{array}{cc|c|c} \frac{1}{2} & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \\ \begin{array}{cccc} \phantom{\frac{1}{2}} & & \downarrow & \downarrow \\ & & a_1 & b_1 \\ & & a_2 & b_2 \end{array} \end{array} \overset{2S_1}{\sim} \left[\begin{array}{cc|c|c} 1 & 2 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right]$$

$$\overset{-2S_1+S_2}{\sim} \left[\begin{array}{cc|c|c} 1 & 2 & 2 & 0 \\ 0 & -2 & -4 & 1 \end{array}\right]$$

$$\overset{-\frac{1}{2}\cdot S_2}{\sim}\left[\begin{array}{cc|c|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & -1/2 \end{array}\right] \overset{-2S_2+S_1}{\sim} \left[\begin{array}{cc|c|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 2 & -\frac{1}{2} \end{array}\right]$$

$$p_1 = a_1 t + a_2 = -2t + 2 \rightarrow \int_0^1 = -t^2 + 2t \Big]_0^1 = 1, \ 0$$

$$p_2 = b_1 t + b_2 = t - \frac{1}{2}$$

$$\left[\begin{array}{cc|c|c} 1 & 2 & 2 & 0 \\ 0 & -2 & -4 & 1 \end{array}\right]$$

$\{ p_1, p_2 \}$ nin duali $\{ \phi_1, \phi_2 \}$ olur.

Örnek: $V = P_2$ üzerinde; t_1, t_2, t_3 farklı reeller olmak üzere:

$$L_i : P_2 \longrightarrow \mathbb{R}$$
$$p(t) \longmapsto p(t_i)$$

şeklinde tanımlanan $\{L_1, L_2, L_3\}$ formları

lin.

1) Lineer Bağımsızdır. ($\{L_1, L_2, L_3\}$ P_2^* ın bazı)

$$a_1 L_1 + a_2 L_2 + a_3 L_3 \equiv 0$$

$$a_1 L_1 + a_2 . L_2 + a_3 L_3 \Big|_{p = 1, t, t^2} \equiv 0$$

$$a_1 L_1(1) + a_2 . L_2(1) + a_3 . L_3(1) \equiv 0$$
$$a_1 . L_1(t) + a_2 L_2(t) + a_3 L_3(t) \equiv 0$$
$$a_1 L_1(t^2) + a_2 L_2(t^2) + a_3 L_3(t^2) \equiv 0$$

$$a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 = 0$$
$$a_1 \cdot t_1 + a_2 \cdot t_2 + a_3 \cdot t_3 = 0$$
$$a_1 \cdot t_1^2 + a_2 \cdot t_2^2 + a_3 \cdot t_3^2 = 0$$

$$\left\{ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ t_1 & t_2 & t_3 & 0 \\ t_1^2 & t_2^2 & t_3^2 & 0 \end{array} \right] \right.$$

Van-der-Monde

$$a_1 L_1(1) + a_2 \cdot L_2(1) + a_3 \cdot L_3(1) \equiv 0$$
$$a_1 \cdot L_1(t) + a_2 L_2(t) + a_3 L_3(t) \equiv 0$$
$$a_1 L_1(t^2) + a_2 L_2(t^2) + a_3 L_3(t^2) \equiv 0$$

t_1, t_2, t_3 birbirinden farklı alındığından; katsayılar matrisinin determinantı 0'dan farklıdır.

$\Rightarrow$ Homojen denkl. sisteminin sadece aşikâr çözümü vardır

$a_1 = a_2 = a_3 = 0 \Rightarrow \{L_1, L_2, L_3\}$ lin. bğ. sızdır. P_2^* için bazdır

② $T = \{L_1, L_2, L_3\}$ bazı P_2 nin hangi bazının dualidir?

P_2 nin bazı $= \{p_1, p_2, p_3\}$ aradığımız baz

ise

$L_1(p_1) = 1 \iff p_1(t_1) = 1, \quad p_1(t_2) = p_1(t_3) = 0$

$L_2(p_2) = 1 \iff p_2(t_2) = 1, \quad p_2(t_1) = p_2(t_3) = 0$

$L_3(p_3) = 1 \iff p_3(t_3) = 1, \quad p_3(t_1) = p_3(t_2) = 0$

$$p_1(t) = \frac{(t-t_2).(t-t_3)}{(t_1-t_2).(t_1-t_3)}, \quad p_2(t) = \frac{(t-t_1).(t-t_3)}{(t_2-t_1).(t_2-t_3)}$$

$$p_3(t) = \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}$$